Exponential Distribution
Modeling Exponential Arrival Times

Gary Schurman, MBE, CFA

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In probability theory, a Poisson process is a stochastic process that counts the number of events in a given time interval. The time between each pair of consecutive events has an exponential distribution with an arrival rate parameter λ and each of these inter-arrival times is assumed to be independent of other inter-arrival times. The probability of an occurrence in a time interval is proportional to the length of the time interval and the probability of simultaneous occurrences equals zero. In this study of inter-arrival times we will use the following hypothetical problem...

Our Hypothetical Problem

ABC Company has built a restaurant on the slopes of an active volcano. If the volcano erupts the restaurant will be destroyed and cannot be rebuilt. Historically this volcano erupts once every 80 years. You are tasked with answering the following questions...

Question 1: What is the probability that the volcano will erupt in the next 10 years?

Question 2: You are modeling expected cash flows via a Monte Carlo simulation. Given that we pull the random number 0.30 from a uniform distribution, when will the volcano erupt for this Monte Carlo trial?

Question 3: What is the mean and variance of the distribution of arrival times?

Modeling Arrival Times

We will define the random variable τ to be the exponentially-distributed arrival time of an event. Given that we are currently standing at time $t = 0$ we want to determine the probability that the event will occur during the time interval $[0, t]$. The domain of the random variable τ is therefore...

$$0 < \tau \leq t \;\text{ if event occurred in time interval}\, [0, t] \;\text{ ...and... }\; \tau > t \;\text{ if event did not occur in time interval}\, [0, t] \tag{1}$$

We will define the variable λ to be the average number of events that we expect to occur during any given time interval $[t, t+1]$. The equation for the probability density function of the exponentially-distributed random variable τ is...

$$f(t) = \lambda\,\mathrm{Exp}\left\{-\lambda\,t\right\} \;\text{ ...when... }\; t \geq 0 \tag{2}$$

Using Equations (1) and (2) above the probability that the event will occur during the time interval $[0, t]$ is...

$$P\left[0 < \tau \leq t\right] = \int_0^t f(s)\,\delta s = \int_0^t \lambda\,\mathrm{Exp}\left\{-\lambda\,s\right\}\delta s = \lambda \int_0^t \mathrm{Exp}\left\{-\lambda\,s\right\}\delta s \tag{3}$$

Using Appendix Equation (24) below the solution to Equation (3) above is...

$$P\left[0 < \tau \leq t\right] = \lambda\,\lambda^{-1}\left(1 - \mathrm{Exp}\left\{-\lambda\,t\right\}\right) = \left(1 - \mathrm{Exp}\left\{-\lambda\,t\right\}\right) \tag{4}$$

Using Equation (4) above the probability that the event will not occur during the time interval $[0, t]$ is...

$$P\bigg[\tau > t\bigg] = 1 - P\bigg[0 < \tau \le t\bigg] = \text{Exp}\bigg\{-\lambda\, t\bigg\} \tag{5}$$

We may want to use random arrival times in Monte Carlo simulations and therefore need a methodology for pulling random arrival times from an exponential distribution. We will define the variable U to be a random number pulled from a uniform distribution with domain $[0, 1]$. The random variable U can be interpreted as the probability that the event will occur during the time interval $[0, t]$. Using Equation (4) above and given the random variable U the equation for the random exponentially-distributed arrival time t is...

$$U = \bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg) \text{ ...such that... } t = -\ln\bigg(1 - U\bigg)\lambda^{-1} \tag{6}$$

First Moment of the Distribution of Arrival Times

Using Equation (2) above the equation for the first moment of the distribution of arrival times is...

$$\mathbb{E}\bigg[\tau\bigg] = \int\limits_0^\infty t\, f(t)\, \delta t = \lambda \int\limits_0^\infty t\, \text{Exp}\bigg\{-\lambda\, t\bigg\}\, \delta t \tag{7}$$

We will make the following definitions for the variable u and its derivative with respect to time...

$$\text{if... } u = t \text{ ...then... } \delta u = \delta t \tag{8}$$

We will make the following definitions for the variable v and its derivative with respect to time...

$$\text{if... } v = -\text{Exp}\bigg\{-\lambda\, t\bigg\} \text{ ...then... } \delta v = \lambda\, \text{Exp}\bigg\{-\lambda\, t\bigg\}\, \delta t \tag{9}$$

Using the definitions in Equations (8) and (9) above we can rewrite Equation (7) via integration by parts as...

$$\mathbb{E}\bigg[\tau\bigg] = \int\limits_0^\infty u\, \delta v = u\, v - \int\limits_0^\infty v\, \delta u = t\, \text{Exp}\bigg\{-\lambda\, t\bigg\} + \int\limits_0^\infty \text{Exp}\bigg\{-\lambda\, t\bigg\}\, \delta t \tag{10}$$

Using Appendix Equation (25) below the solution to Equation (10) above is...

$$\begin{aligned}\mathbb{E}\bigg[\tau\bigg] &= t\, \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg[_{t=0}^{t=\infty} + \lambda^{-1}\bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg)\bigg[_{t=0}^{t=\infty} \\ &= t \times 0 + \lambda^{-1}\bigg(1 - 0\bigg) \\ &= \lambda^{-1}\end{aligned} \tag{11}$$

Second Moment of the Distribution of Arrival Times

Using Equation (2) above the equation for the second moment of the distribution of arrival times is...

$$\mathbb{E}\bigg[\tau^2\bigg] = \int\limits_0^\infty t^2\, f(t)\, \delta t = \lambda \int\limits_0^\infty t^2\, \text{Exp}\bigg\{-\lambda\, t\bigg\}\, \delta t \tag{12}$$

We will make the following definitions for the variable u and its derivative with respect to time...

$$\text{if... } u = t^2 \text{ ...then... } \delta u = 2\, t\, \delta t \tag{13}$$

We will make the following definitions for the variable v and its derivative with respect to time...

$$\text{if... } v = -\text{Exp}\bigg\{-\lambda\, t\bigg\} \text{ ...then... } \delta v = \lambda\, \text{Exp}\bigg\{-\lambda\, t\bigg\}\, \delta t \tag{14}$$

Using the definitions in Equations (13) and (14) above we can rewrite Equation (12) via integration by parts as...

$$\mathbb{E}\bigg[\tau^2\bigg] = \int\limits_0^\infty u\,\delta v = u\,v - \int\limits_0^\infty v\,\delta u = t^2\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} + 2\int\limits_0^\infty t\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,\delta t \tag{15}$$

Using Equation (11) above we can rewrite Equation (15) above as...

$$\mathbb{E}\bigg[\tau^2\bigg] = t^2\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} + 2\,\lambda^{-1}\,\mathbb{E}\bigg[\tau\bigg] \tag{16}$$

The solution to Equation (16) above is...

$$\begin{aligned}\mathbb{E}\bigg[\tau^2\bigg] &= t^2\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg[_{t=0}^{t=\infty} + 2\,\lambda^{-1}\lambda^{-1}\\ &= \bigg((\infty^2\times 0) - (0^2\times 1)\bigg) + 2\,\lambda^{-2}\\ &= 2\,\lambda^{-2}\end{aligned} \tag{17}$$

Mean and Variance of the Distribution of Arrival Times

Using Equation (11) above the equation for the mean of the distribution of arrival times is...

$$\text{mean} = \mathbb{E}\bigg[\tau\bigg] = \lambda^{-1} \tag{18}$$

Using Equations (11) and (17) above the equation for the variance of the distribution of arrival times is...

$$\text{variance} = \mathbb{E}\bigg[\tau^2\bigg] - \bigg(\mathbb{E}\bigg[\tau\bigg]\bigg)^2 = 2\,\lambda^{-2} - \lambda^{-2} = \lambda^{-2} \tag{19}$$

The Answers To Our Hypothetical Problem

To answer the questions posed to us we must first calculate the variable λ, which is the event arrival rate. The equation for λ is...

$$\lambda = \frac{1}{80} = 0.0125 \tag{20}$$

Question 1: Using Equation (4) above the probability that the volcano will erupt within the next 10 years is...

$$P\bigg[0 < \tau \leq 10\bigg] = \bigg(1 - \mathrm{Exp}\bigg\{-0.0125\times 10\bigg\}\bigg) = 0.1175 \tag{21}$$

Question 2: Using Equation (6) above and given that we pulled a random number of 0.30 from a uniform distribution the arrival time for our Monte Carlo simulation is...

$$t = -\ln\bigg(1 - U\bigg)\lambda^{-1} = -\ln\bigg(1 - 0.30\bigg)\times\frac{1}{0.0125} = 28.53\,\text{years} \tag{22}$$

Question 3: Using Equations (18) and (19) above the mean and variance of the distribution of arrival times is...

$$\text{mean} = \lambda^{-1} = \frac{1}{0.0125} = 80\,\text{years}\ \text{...and...}\ \text{variance} = \lambda^{-2} = \frac{1}{0.0125^2} = 6400\,\text{years} \tag{23}$$

Appendix

B. Integral solution for the equation...

$$\int\limits_0^t \mathrm{Exp}\bigg\{-\lambda\,s\bigg\}\,\delta s = \mathrm{Exp}\bigg\{-\lambda\,s\bigg\}\bigg[_0^t = \mathrm{Exp}\bigg\{-\lambda\times\infty\bigg\} - \mathrm{Exp}\bigg\{-\lambda\times 0\bigg\} = -1 \tag{24}$$

B. Evaluate the integral solution...

$$\int\limits_0^\infty \mathrm{Exp}\bigg\{-\lambda\,s\bigg\}\,\delta s\ \text{...where...}\ \mathrm{Exp}\bigg\{-\lambda\times 0\bigg\} = 1\ \text{...and...}\ \mathrm{Exp}\bigg\{-\lambda\times\infty\bigg\} = 0 \tag{25}$$